GLUING BOREL-SMITH FUNCTIONS AND THE GROUP OF ENDO-TRIVIAL MODULES

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Abstract. The aim of this paper is to describe the group of endo-trivial modules for a p-group P, in terms of the obstruction group for the gluing problem of Borel-Smith functions. Explicitly, we shall prove that there is a split exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow T(P) \longrightarrow \mathrm{Obs}(C_b(P)) \longrightarrow 0$$

of abelian groups where $T(P)$ is the endo-trivial group of P, and $C_b(P)$ is the group of Borel-Smith functions on P. As a consequence, we obtain a set of generators of the group $T(P)$ that coincides with the relative syzygies found by Alperin. In order to prove the result, we solve gluing problems for the functor B^* of super class functions, the functor $\mathcal{R}^*_{\mathbb{Q}}$ of rational class functions and the functor C_b of Borel-Smith functions.

Contents

1. Introduction

In [13] and [14] Dade introduced the class of endo-permutation modules for p-groups together with a certain sub-class of endo-trivial modules and classified them for abelian p-groups. He also raised the question of the classification of endo-permutation modules for all p-groups. A systematic study of the endo-permutation modules is done by considering the group of the so-called capped endo-permutation modules, now called the Dade group and denoted by $D(P)$ for the group P. The *subgroup* of endo-trivial modules is denoted by $T(P)$.

The classification problem for endo-permutation modules is solved, after 25 years since it was raised, in several steps. One of the most important steps in this direction is the description of the group $T(P)$ of the endo-trivial modules by Carlson and Thévenaz in [11] and [10]. The final classification was done by Bouc in [4] where he

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used the description of the group $T(P)$ and the theory of biset functors, introduced and developed by him. See [2] for the theory of biset functors.

This paper is a contribution to the description of the group $T(P)$. Regarding the original description, in [1], Alperin constructed a certain set of relative syzygies associated to a finite p-group P and proveed that they generate a subgroup of finite index in the group $T(P)$ of all endo-trivial modules. Later Carlson and Thévenaz in [10] showed that this subgroup is equal to the torsion free part of the group $T(P)$. Also in [11], they describe the torsion part of the group $T(P)$ completing the classification of endo-trivial modules for p-groups. The above set of generators is also constructed by Carlson in [9] using cohomology. We refer to [9], [10] or [18], for further details.

The aim of this paper is to give an another construction of the above set of relative syzygies and to prove that they generate the group $T(P)$. To construct the relative syzygies we relate the group of endo-trivial modules to the group of obstructions for the gluing problem for Borel-Smith functions and hence use the theory of biset functors for the construction. We do use the classification theorem of Carlson and Thévenaz regarding the torsion part of the group $T(P)$ but we do not need the full classification of endo-permutation (or endo-trivial) modules. See the end of the paper for a revision of the results that we have used throughout the paper.

Returning to the new construction of the relative syzygies, the gluing problem for a biset functor F at a finite group G is the problem of finding an element $f \in F(G)$ associated to a given sequence $(f_H)_{1<H\leq G}$ where f_H is an element in $F(N_G(H)/H)$, if it exists, see Section 2 for details. This problem is first considered by Bouc and Thévenaz [7] for the functor of torsion endo-permutation modules and later by Bouc [5] for arbitrary endo-permutation modules, both for p-groups with p odd. The first problem is solved whereas the latter is incomplete. In this paper, we consider the gluing problem for three more biset functors, namely for the functor of super class functions, for the functor of rational class functions and for the functor of Borel-Smith functions.

The relevance of the gluing problem of the functor of Borel-Smith functions to the group $T(P)$ is given by our main result, Theorem 6.1, which proves that there is a split exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow T(P) \rightarrow \mathrm{Obs}(C_b(P)) \rightarrow 0 \qquad (*)$$

where $T(P)$ is the endo-trivial group of P, and $C_b(P)$ is the group of Borel-Smith functions on P.

To obtain this exact sequence, we use the exact sequences

$$0 \longrightarrow C_b \longrightarrow B^* \longrightarrow D^\Omega \longrightarrow 0$$

and

$$0 \longrightarrow C_b \longrightarrow \mathcal{R}^*_{\mathbb{Q}} \longrightarrow D^\Omega_t \longrightarrow 0$$

of biset functors of Theorem 1.2 and Theorem 1.3 of [8], and the result of Puig which determines the kernel of the map $D(P) \to \varprojlim_{1<Q\leq P} D(N_P(Q)/Q)$ as the endo-trivial group $T(P)$. Finally we need the solutions of the other to gluing problem mentioned above.

Now the exact sequence $(*)$ is the kernel-cokernel sequence of the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C_b(P) \arrow[r] \arrow[d, "r_P^{C_b}"'] & B^*(P) \arrow[r, "\Psi_P"] \arrow[d, "r_P^{B^*}"'] & D^\Omega(P) \arrow[r] \arrow[d, "r_P^D"'] & 0 \\ 0 \arrow[r] & \varprojlim_{1<Q\le P} C_b(W_P(Q)) \arrow[r] & \varprojlim_{1<Q\le P} B^*(W_P(Q)) \arrow[r, "\tilde{\Psi}_P"] & \varprojlim_{1<Q\le P} D^\Omega(W_P(Q)) & \end{tikzcd}$$

with exact rows, where we put $W_P(Q) := N_P(Q)/Q$ and the splitting can be constructed by considering a similar commutative diagram with first row equal to the exact sequence of Theorem 1.3 in [8]. See Section 6 for further details.

Regarding the gluing problems, the solution for the gluing problem for super class functions is in Section 3. We prove, as Theorem 3.1, that there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow B^*(G) \longrightarrow \varprojlim_{1<H\le G} B^*(N_G(H)/H) \longrightarrow 0$$

of abelian groups. In particular, the gluing problem for super class functions always has infinitely many solutions. Moreover, it is clear from the proof of this theorem, given in Section 3, that solutions to a given gluing data differ only by their evaluations at the trivial subgroup.

The solution of the problem for rational class functions for a p-group P of rank at least 2 is given by the exact sequence

$$0 \longrightarrow \mathcal{R}^*_{\mathbb{Q}}(P) \xrightarrow{r_P} \varprojlim_{1<Q\le P} \mathcal{R}^*_{\mathbb{Q}}(N_P(Q)/Q) \xrightarrow{\tilde{d}_P} \tilde{H}^0(\mathcal{A}_{\ge 2}(P), p\mathbb{Z})^P \longrightarrow 0$$

of abelian groups. See Section 4 for the details of the notation. Note that if the rank of P is 1 then $\mathcal{R}^*_{\mathbb{Q}} = B^*$ and the problem is reduced to the previous case.

The gluing problem for Borel-Smith functions is solved in Section 5 where we prove that for a p-group of rank at least 2, there is an exact sequence

$$0 \longrightarrow C_b(P) \xrightarrow{r_P} \varprojlim_{1<Q\le P} C_b(N_P(Q)/Q) \xrightarrow{\tilde{d}_P} \tilde{H}^0_b(\mathcal{A}_{\ge 2}(P), p\mathbb{Z})^P \longrightarrow 0$$

of abelian groups.

When the group P has rank 1, we obtain the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow C_b(P) \xrightarrow{r_P} \varprojlim_{1<Q\le P} C_b(N_P(Q)/Q) \longrightarrow \mathbb{Z}/m_P\mathbb{Z} \longrightarrow 0$$

of abelian groups where $m_P = 2$ if P is cyclic and $m_P = 4$ if P is generalized quaternion.

Finally, to simplify our notation, we introduce a general setup for the gluing problem for an arbitrary biset functor in Section 2.

2. Gluing problem in general

Let F be a biset functor and G be a finite group. A *gluing data* for $F(G)$ is a sequence $(f_H)_{1<H\le G}$ of elements $f_H \in F(N_G(H)/H)$ satisfying the following compatibility conditions:

(i) (Conjugation invariance) For any $g \in G$ and $H \le G$, we have

$${}^g f_H = f_{{}^g H}.$$

(ii) (Destriction invariance) For any pair $H \trianglelefteq K$ of subgroups of G, we have

$$\mathrm{Des}^{N_G(H)/H}_{N_G(H,K)/K} f_H = \mathrm{Res}^{N_G(K)/K}_{N_G(H,K)/K} f_K.$$

Here we use the notation $\mathrm{Des}^{N_G(H)/H}_{N_G(H,K)/K}$ for the composition of deflation and restriction maps. Note that the same composition is written as Defres in [5]. We call the composite map Des the *destriction map*.

Now the *gluing problem* for the biset functor F at G is the problem of finding an element $f \in F(G)$ such that for any non-trivial subgroup H of G, we have

$$\mathrm{Des}^{G}_{N_G(H)/H} f = f_H.$$

If such an element exists, we call it a *solution* to the given gluing data. Following Bouc and Thévenaz [7], we denote by

$$\varprojlim_{1<H\leq G} F(N_G(H)/H)$$

the set of all gluing data for $F(G)$.

It is clear that given a biset functor F, there is a homomorphism

$$r_G := r_G^F : F(G) \to \varprojlim_{1<H\leq G} F(N_G(H)/H)$$

given by associating an element $f \in F(G)$ to the sequence $(\mathrm{Des}^G_{N_G(H)/H} f)_{1<H\leq G}$. Now the gluing problem is asking if this map is surjective, cf. [7, Section 2]. A *complete solution to the gluing problem* is an exact sequence

$$0 \longrightarrow \mathrm{Ker}(r_G) \longrightarrow F(G) \xrightarrow{r_G} \varprojlim_{1<H\leq G} F(N_G(H)/H) \longrightarrow \mathrm{Obs}(F(G)) \longrightarrow 0$$

where $\mathrm{Obs}(F(G))$ denotes the cokernel of the map r_G and it is the group of obstructions for the gluing data to have a solution.

An easy (but very useful) observation about the general gluing problem is that we can lift any short exact sequence of biset functors to an exact sequence of the corresponding groups of gluing data and hence relate the obstruction groups for the solutions. This can be done as follows.

Proposition 2.1. *Let*

$$0 \longrightarrow M \longrightarrow F \xrightarrow{\Phi} N \longrightarrow 0$$

be an exact sequence of biset functors. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & M(G) \arrow[r] \arrow[d, "r_G^M"'] & F(G) \arrow[r, "\Phi^G"] \arrow[d, "r_G^F"'] & N(G) \arrow[r] \arrow[d, "r_G^N"'] & 0 \\
0 \arrow[r] & \varprojlim_{1<H\leq G} M(W_G(H)) \arrow[r] & \varprojlim_{1<H\leq H} F(W_G(H)) \arrow[r, "\tilde{\Phi}^G"] & \varprojlim_{1<H\leq G} N(W_G(H)) &
\end{tikzcd}$$

commutes and has exact rows. Here $\tilde{\Phi}^G$ is defined as the sequence $(\Phi^{W_G(H)})_{1<H\leq G}$ of maps and we put $W_G(H) := N_G(H)/H$.

The proof of the above proposition is trivial since Φ is a morphism of biset functors. As a corollary, applying the snake lemma, we get an exact sequence of abelian groups

$$0 \longrightarrow K(r_G^M) \longrightarrow K(r_G^F) \longrightarrow K(r_G^N) \longrightarrow \mathrm{Obs}(M(G)) \longrightarrow \mathrm{Obs}(F(G)) \longrightarrow \mathrm{Obs}(N(G))$$

where we put $K(r_G^F) := \mathrm{Ker}(r_G^F)$.

Remark 1. *Given a biset functor F, it is clear that the functor associating a group G to its gluing data $\varprojlim_{1<H\leq G} F(W_G(H))$ is not a biset functor. Nevertheless this functor has a structure of a destriction functor. By a* destriction functor*, we mean a biset functor without induction and inflation and in this sense, they are generalizations of restriction functors from the context of Mackey functors, see* [12] *for details. Indeed, the maps*

$$\mathrm{Res}^G_K : \varprojlim_{1<H\leq G} F(W_G(H)) \to \varprojlim_{1<H\leq K} F(W_K(H))$$

for any pair of groups $K \leq G$ are defined in [7] *and in a similar way, we can define deflation map and transport of structure by a group isomorphism. We skip the details of the definition of these maps.*

We end the section by the following generalization of a result of [7], which we will use later. It shows that the gluing problem for an arbitrary biset functor F at an elementary abelian group is always solvable. We skip the proof since it is almost identical to the proof of Lemma 2.2 of [7].

Lemma 2.2 (Bouc-Thévenaz)**.** *Let E be an elementary abelian group. Then the map*

$$r_E^F : F(E) \to \varprojlim_{1<F\leq E} F(E/F)$$

is surjective with a section given by

$$(u_F)_{1<F\leq E} \mapsto -\sum_{1<F\leq E} \mu(1,F)\mathrm{Inf}^E_{E/F} u_F.$$

where μ is the Mobius function of the poset of subgroups of E.

3. Gluing super class functions

Let G be a finite group. We denote by $B(G)$ the Burnside ring of G. We recall that the Burnside ring of a finite group G is the Grothendieck ring of the category of finite G-sets and as a free abelian group, it is generated by the set of conjugacy classes of subgroups of G. By $B^*(G)$ we denote the $\mathbb{Z}$-dual of the Burnside ring. It is well-known that the group $B^*(G)$ can be identified with the ring of super class functions $S(G) \to \mathbb{Z}$. Here $S(G)$ is the set of all subgroups of G and a *super class function* is a function which is constant on the conjugacy classes of subgroups of G. The functor sending a group G to the dual Burnside ring $B^*(G)$ is a biset functor with the usual actions of bisets. We refer to [8] for further details.

The following theorem shows that the answer to the the gluing problem for super class functions is always affirmative.

Theorem 3.1. *There is an exact sequence of abelian groups*

$$0 \longrightarrow \mathbb{Z} \longrightarrow B^*(G) \xrightarrow{r_G} \varprojlim_{1<H\leq G} B^*(N_G(H)/H) \longrightarrow 0$$

where the map r_G is as defined in the previous section.

Proof. The proof has two parts. First we prove that the kernel of r_G is isomorphic to $\mathbb{Z}$ and second we show that the map r_G is surjective.

For the first part, let f be a super class function with $r_G(f) = 0$. By the definition of the map r_G, we have

$$\mathrm{Des}^G_{N_G(H)/H} f = 0$$

for any non-trivial subgroup H of G. But by the definition of the action of destriction on B^*, we have

$$0 = (\mathrm{Des}^G_{N_G(H)/H} f)(L/H) = f(L)$$

for any $L/H \le N_G(H)/H$. Therefore the super class function f is zero at all non-trivial subgroups of G. Also it is clear that its value at the trivial subgroup can be chosen freely. Therefore we have proved that a super class function f is in the kernel of r_G if and only if it is zero at any non-trivial subgroup of G, as required.

To prove that r_G is surjective, let $(f_H)_{1<H\le G}$ be a gluing data. We claim that the super class function $f \in B^*(G)$ defined by $f(H) = f_H(H/H)$ and $f(1) = 0$ is a pre-image of the given gluing data. Firstly, f is a super class function since for any $H \le G$ and $g \in G$, we have

$$f({}^xH) = f_{{}^xH}({}^xH/{}^xH) = ({}^xf_H)({}^xH/{}^xH) = {}^x(f_H(H/H)) = f_H(H/H) = f(H).$$

Here we use the conjugation invariance of the gluing data. Now it remains to show that for any $1 < H \le G$, we have

$$\mathrm{Des}^G_{N_G(H)/H} f = f_H.$$

Notice that by the destriction invariance of the gluing data, for any pair $H \trianglelefteq K$, we have

$$f_H(K/H) = f_K(K/K).$$

Indeed we have

$$f_H(K/H) = (\mathrm{Des}^{N_G(H)/H}_{N_G(H,K)/K} f_H)(K/K) = (\mathrm{Res}^{N_G(K)/K}_{N_G(H,K)/K} f_K)(K/K) = f_K(K/K).$$

Now it follows that

$$(\mathrm{Des}^G_{N_G(H)/H} f)(K/H) = f(K) = f_K(K/K) = f_H(K/H),$$

for any $K/H \le N_G(H)/H$, as required. □

Corollary 3.2. *A gluing problem for super class functions at a finite group G has infinitely many solutions.*

Proof. Let $(f_H)_{1<H\le G}$ be a gluing data. Then the super class function f defined by putting $f(H) := f_H(H/H)$ if $H \ne 1$ and $f(G/1) = a$ is a solution of the gluing data for any $a \in \mathbb{Z}$. □

4. Gluing rational class functions for p-groups

Let p be a prime number and P be a finite p-group. Denote by $\mathcal{R}_{\mathbb{Q}}(P)$ the rational character ring of P. This is the Grothendieck ring of the category of $\mathbb{Q}P$-modules. With the usual definitions of induction, restriction, inflation, deflation and transport of structure, the assignment $P \mapsto \mathcal{R}_{\mathbb{Q}}(P)$ is a biset functor. We denote the $\mathbb{Z}$-dual of this biset functor by $\mathcal{R}^*_{\mathbb{Q}}$. We call an element of $\mathcal{R}^*_{\mathbb{Q}}(P)$ a *rational class function* for P. We refer to [2] for further details.

Recall that by a theorem of Ritter and Segal, the linearization map lin: $B(P) \to \mathcal{R}_{\mathbb{Q}}(P)$, associating a finite P-set X to the permutation $\mathbb{Q}P$-module $\mathbb{Q}X$ with basis

X, is surjective. Hence its transpose gives an injective map $\mathcal{R}^*_{\mathbb{Q}}(P) \to B^*(P)$, cf [3]. Note also that the linearization map is a morphism of biset functors. Therefore the biset functor $\mathcal{R}^*_{\mathbb{Q}}$ can be identified with a subfunctor of the dual B^* of the Burnside functor B, via the transpose of the linearization map.

Therefore this inclusion gives rise to an inclusion of the group of gluing data for $\mathcal{R}^*_{\mathbb{Q}}$ to that of B^*. Hence by Corollary 3.2, a gluing data for $\mathcal{R}^*_{\mathbb{Q}}$ can always be glued to a super class function. Thus we need to determine when the solution is a rational class function. We recall the following result from [8, Lemma 4.2]. In the following, we denote the unique group of order p by C_p.

Lemma 4.1 (Bouc-Yalçın [8])**.** *Let f be a super class function for the p-group P. Then f is a rational class function if and only if for any subquotient T/S of P, isomorphic to $C_p \times C_p$, the condition*

$$f(P/S) - f(P/T) = \sum_{S<X<T} (f(P/X) - f(P/T)) \qquad (*).$$

is satisfied.

Now let $(f_Q)_{1<Q\leq P}$ be a gluing data for $\mathcal{R}^*_{\mathbb{Q}}(P)$ and let f be a solution of the data in $B^*(P)$. Also let T/S be a subquotient isomorphic to $C_p \times C_p$. We claim that the equality

$$f(P/Y) = f_S((N_P(S)/S)/(Y/S))$$

is satisfied for any $S \trianglelefteq Y \leq P$. Indeed, by the definition of f, we have

$$f(P/Y) = f_Y((N_P(Y)/Y)/(Y/Y))$$

and by the destriction invariance of the gluing data we have

$$f_Y((N_P(Y)/Y)/(Y/Y)) = f_S((N_P(S)/S)/(Y/S)).$$

Here we apply the condition with $S \trianglelefteq Y$. Therefore we can write the above equality $(*)$ as

$$f_S(N_P(S)/S) - f_S(N_P(S)/S) = \sum_{S<X<T} (f_S(N_P(S)/X) - f_S(N_P(S)/T)).$$

Here we shorten $(N_P(S)/S)/(Y/S)$ to $N_P(S)/Y$. Now if S is not the trivial subgroup, this condition is satisfied since $f_S \in \mathcal{R}^*_{\mathbb{Q}}(N_P(S)/S)$. Thus we have proved the following lemma.

Lemma 4.2. *Let $(f_Q)_{1<Q\leq P}$ be a gluing data for $\mathcal{R}^*_{\mathbb{Q}}(P)$ and let f be a solution of the data in $B^*(P)$. Then f is a rational class function if and only if for any elementary abelian subgroup E of P of rank 2, the condition $(*)$ of Lemma 4.1 is satisfied.*

Our next aim is to determine the group of obstructions for a gluing data for $\mathcal{R}^*_{\mathbb{Q}}$. First note that if P has rank 1, then the condition of the above Lemma is trivially satisfied and hence we get that $\mathcal{R}^*_{\mathbb{Q}} = B^*$. Therefore we reduced to the previous case.

Therefore we assume that P has rank at least 2, hence P is neither cyclic nor generalized quaternion. Then by the previous lemma, a solution f of the gluing data $(f_Q)_{1<Q\leq P}$ in the dual Burnside group $B^*(P)$ is not a rational class function if there is an elementary abelian subgroup E of P of rank 2 such that the condition $(*)$ of Lemma 4.1 is not satisfied. Notice that the condition $(*)$ actually determines

the value at the trivial group of the solution, since for E, the condition $(*)$ can equivalently be written as

$$f(P/1) = \sum_{1<X<T} f(P/X) - pf(P/T)$$

and the values $f(P/X)$ and $f(P/T)$ are determined by the gluing data. Hence the obstruction group should check whether the value at the trivial subgroup is well-defined, that is, if for all possible choices of the subgroup E, the value $f(P/1)$ is constant.

To determine the obstruction group, first we introduce a notation. We denote by $\mathcal{A}_{\geq 2}(P)$ the P-poset of all elementary abelian subgroups of P of rank at least 2, and where P acts by conjugation. It is clear that any maximal elementary abelian subgroup of P of rank 2 is an isolated vertex in $\mathcal{A}_{\geq 2}(P)$. It is also easy to see that the rest of the elementary abelian subgroups lie in a connected component, which is called the *big component* of $\mathcal{A}_{\geq 2}(P)$, see [10, Lemma 2.1]. We refer to Lemma 2.2 in [10] and Lemma 3.1 in [7] for further properties of this poset.

Now following Bouc and Thévenaz [7], we write $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^P$ for the group of P-invariant functions f from $\mathcal{A}_{\geq 2}$ to the additive group $\mathbb{Z}$ of integers, satisfying the condition that $f(E) = f(F)$ if $E \leq F$, modulo the constant functions. By Lemma 3.1 in [7], the group $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^P$ is generated by the characteristic functions of the classes of maximal elementary abelian subgroups of P of rank 2. Now we define a map, denoted by $\tilde{d}_P$, cf. [7, Section 3],

$$\tilde{d}_P : \varprojlim_{1<Q\leq P} \mathcal{R}^*_{\mathbb{Q}}(N_P(Q)/Q) \to \tilde{H}^0(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^P$$

by associating the data $(f_Q)_{1<Q\leq P}$ to the class of the function $\tilde{f}$ which maps a subgroup E of order p^2 to $\sum_{1<X<E} f(P/X) - pf(P/E)$ and a subgroup F of order more than p^2 to $\tilde{f}(E)$ for some subgroup E of F of order p^2.

Here we denote by f the solution of $(f_Q)_{1<Q\leq P}$ in $B^*(P)$. Here after, we write $[\phi]$ to denote the image of $\phi \in H^0(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^P$ in the quotient group $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^P$.

Lemma 4.3. *The map*

$$\tilde{d}_P : \varprojlim_{1<Q\leq P} \mathcal{R}^*_{\mathbb{Q}}(N_P(Q)/Q) \to \tilde{H}^0(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^P$$

as defined above is a group homomorphism.

Proof. First we need to check that the map is well-defined, that is, we need to check that for any $(f_Q)_{1<Q\leq P} \in \varprojlim_{1<Q\leq P} \mathcal{R}^*_{\mathbb{Q}}(N_P(Q)/Q)$, the function $\tilde{f} := \tilde{d}_P((f_Q)_{1<Q\leq P})$ is in $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^P$. First, it is clear, by conjugation invariance of the gluing data, that the resulting function is P-invariant. To prove the condition that $f(E) = f(F)$ if $E \leq F$, note that we only need to check this condition for the big component of $\mathcal{A}_{\geq 2}(P)$, which contains a normal elementary abelian p-subgroup. By Lemma 2.1 in [10], this component is unique. So it suffices to show that the function $\tilde{f}$ restricted to the set of subgroups of order p^2 in the big component is constant. Let E be a maximal element of the big component. Then by Lemma 2.2, the gluing data $\mathrm{Res}^P_E((f_Q)_{1<Q\leq P})$ has a solution which implies, by Lemma 4.2, that for any pair of subgroups T, S of rank 2 of E, we have $\tilde{f}(T) = \tilde{f}(S)$. Now the result follows since for any pair T, S of subgroups of rank 2 in the big component,

TS is elementary abelian and contains both T and S, as required. The claim that $\tilde{d}_P$ is a group homomorphism is trivial. □

Unfortunately $\tilde{d}_P$ is not surjective but it is possible to determine the image. First of all, if $\mathcal{A}_{\geq 2}(P)$ is connected, then the group $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^P$ vanishes and hence the map $\tilde{d}_P$ becomes the zero map. Thus, without loss of generality, we can assume that the poset $\mathcal{A}_{\geq 2}$ contains at least two connected components.

Then by [7, Lemma 3.1], there is a unique central subgroup Z of P of order p contained in each maximal elementary abelian subgroup of rank 2 and moreover given such a group E, all the other subgroups of order p are P-conjugate.

On the other hand, by [16], if p is odd, and if the rank of P is at least 3, then by Proposition 2.5 in [15], there is a unique normal elementary abelian subgroup E_0 of P of rank 2 which is non-central and characteristic, which implies that E_0 contains the subgroup Z and all the other subgroups of order p are P-conjugate.

Similarly, if $p = 2$ and the rank of P is at least 3, then there is a non-maximal normal elementary abelian subgroup E_0 of rank 2 whose two non-central elements of order 2 are P-conjugate. (Note that in this case, as in the previous one, we have $|P/C_P(E_0)| = 2$.)

Thus if E is either an isolated vertex in $\mathcal{A}_{\geq 2}(P)$ or is equal to E_0, defined above, let Y be a subgroup of order p in E different from Z. Then we have

$$\sum_{1<X<E} f(P/X) - pf(P/E) = f(P/Z) + p(f(P/Y) - f(P/E))$$

where we use the conjugation invariance of the gluing data to collect together all the terms $f(P/X)$ with non-central X. Now it is clear from this equation that

$$[E \mapsto \sum_{1<X<E} f(P/X) - pf(P/E)] = [E \mapsto p(f(P/Y) - f(P/E))]$$

for some non-central subgroup Y of order p in E. Therefore the image of $\tilde{d}_P$ is contained in the subgroup $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P$ of $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^P$ consisting of functions having values in $p\mathbb{Z}$.

Next we show that this subgroup is equal to the image of $\tilde{d}_P$. Note that the subgroup $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P$ is generated by functions $p\chi_E$ as E runs over a complete set of representatives of conjugacy classes of maximal elementary abelian subgroups of rank 2 and where χ_E is the characteristic function of the class of E.

Let E be a maximal elementary abelian subgroup of P of rank 2 and let χ_E be the characteristic function of the conjugacy class of E. We define a sequence $(f_Q)_{1<Q\leq P}$ as follows. If Q is not P-conjugate to Y, then put $f_Q = 0$. If Q is conjugate to Y, then define f_Q as the super class function for $N_P(Q)/Q$ given by $f_Q(N_P(Q)/T) = \delta_{T,Q}$, where $\delta_{?,?}$ is the Kronecker's delta. We claim that the sequence is a gluing data for $\mathcal{R}^*_{\mathbb{Q}}(P)$ and its image is equal to $p\chi_E$. The first claim is almost trivial since most of the f_Q's are zero and the group $N_P(Y)/Y$ is cyclic or generalized quaternion by [7, Lemma 3.1]. So we need to check that $\tilde{d}_P((f_Q)_{1<Q\leq P}) = p\chi_E$ but this follows easily from the definition of the map $\tilde{d}_P$.

The above arguments prove that there is a surjective group homomorphism

$$\tilde{d}_P : \varprojlim_{1<Q\leq P} \mathcal{R}^*_{\mathbb{Q}}(N_P(Q)/Q) \to \tilde{H}^0(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P.$$

Next we show that the kernel of this map is exactly the image of the map r_P. For this aim, let f be a rational class function and $(f_Q)_{1<Q\leq P}$ be its image under r_P.

Then

$$\tilde{d}_P((f_Q)_{1<Q\leq P})(E) = p(f(P/Y) - f(P/E))$$

where E and Y are as above. But by Lemma 4.1, we have $p(f(P/Y) - f(P/E)) = f(P/1) - f(P/Z)$ where Z is as before. Since Z is contained in any maximal elementary abelian subgroup of rank 2, the right hand side of the above equality is independent of the choice of E, that is, $\tilde{d}_P(r_P(f))$ is constant. Hence $\tilde{d}_P(r_P(f)) = 0$ as required. Similarly, if $(f_Q)_{1<Q\leq Q}$ is a sequence in the kernel of $\tilde{d}_P$, then a solution of $(f_Q)_{1<Q\leq P}$ in $B^*(P)$ can be chosen to be a rational class function, by choosing the value at the trivial subgroup to be the constant that is determined by the gluing data under the map $\tilde{d}_P$ (so that the condition of Lemma 4.2 is satisfied).

We remark that a similar argument shows that the map r_P is injective. Indeed a rational class function f is in the kernel of r_P if and only if it is zero at any nontrivial subgroup of P. But by Lemma 4.2, the value at the trivial group is uniquely determined by the values at nontrivial subgroups. Hence f should be the zero function, as required. With this remark, we have finished the proof of the following theorem.

Theorem 4.4. *Let p be a prime number and P be a p-group of rank at least 2. Then there is an exact sequence of abelian groups*

$$0 \longrightarrow \mathcal{R}^*_{\mathbb{Q}}(P) \xrightarrow{r_P} \varprojlim_{1<Q\leq P} \mathcal{R}^*_{\mathbb{Q}}(N_P(Q)/Q) \xrightarrow{\tilde{d}_P} \tilde{H}^0(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P \longrightarrow 0$$

where the maps r_P and $\tilde{d}_P$ are as defined above.

Note that when the poset $\mathcal{A}_{\geq 2}(P)$ is connected, the group $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P$ becomes the trivial group and we get that the gluing problem always has a unique solution, which proves the following corollary.

Corollary 4.5. *Let P be a p-group of rank at least 2 such that the poset $\mathcal{A}_{\geq 2}(P)$ is connected. Then the map*

$$r_P^{\mathcal{R}^*_{\mathbb{Q}}} : \mathcal{R}^*_{\mathbb{Q}}(P) \to \varprojlim_{1<Q\leq P} \mathcal{R}^*_{\mathbb{Q}}(N_P(Q)/Q)$$

is an isomorphism.

Finally, the following corollary is an easy consequence of the above proof, completing the solution of the gluing problem for rational class functions for p-groups.

Corollary 4.6. *Let P be a p-group of rank 1. Then there is an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{R}^*_{\mathbb{Q}}(P) \xrightarrow{r_P} \varprojlim_{1<Q\leq P} \mathcal{R}^*_{\mathbb{Q}}(N_P(Q)/Q) \longrightarrow 0$$

of abelian groups.

5. Gluing Borel-Smith functions

Let p be a prime number and P be a p-group. As in Section 3, let $B^*(P)$ denotes the group of super class functions for P. Following tom Dieck [19], we call a super class function $f \in B^*(P)$ a *Borel-Smith function* if the following conditions are satisfied:

1. If p is odd, then for any subquotient T/S of P, of order p, the number $f(T) - f(S)$ is even.

(2) If $p = 2$, then for any sequence of subgroups $H \trianglelefteq K \trianglelefteq L \leq N_P(H)$, with $|K : H| = 2$, the value $f(K) - f(H)$ is even if L/K is cyclic of order 4 and is divisible by 4 if L/K is quaternion of order 8.

(3) For any elementary abelian subquotient T/S of P, of rank 2, the equality

$$f(S) - f(T) = \sum_{S<X<T} (f(X) - f(T))$$

holds.

It is proved in [8] that the subgroups $C_b(Q)$ of Borel-Smith functions as Q runs over all p-groups together with the induced actions of bisets is a subfunctor of the biset functor B^*.

To solve the gluing problem for Borel-Smith functions, we introduce the following group. Define the group

$$\tilde{H}^0_b(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P$$

as the subgroup of $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P$ generated by the elements $pn_E\chi_E$ where E runs over a complete set of representatives of conjugacy classes of maximal elementary abelian subgroups of P of rank 2 and for such a subgroup E, the integer n_E is given by

$$n_E = \begin{cases} 1, & \text{if } |C_P(Y) : Y| = 2; \\ 2, & \text{if } C_P(Y)/Y \text{ is cyclic of order} \geq 3; \\ 4, & \text{if } C_P(Y)/Y \text{ is generalized quaternion} \end{cases}$$

where Y is a non-central proper subgroup of E and recall that by Lemma 2.2 in [10], the group $C_P(Y)/Y$ is either cyclic or generalized quaternion.

Now we define a map

$$\tilde{d}_P : \varprojlim_{1<Q\leq P} C_b(N_P(Q)/Q) \to \tilde{H}^0_b(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P$$

by associating a given sequence $(b_Q)_{1<Q\leq P}$ to the class of the function $E \mapsto p(b_S(W_P(S)/(S/S)) - b_S(W_P(S)/(E/S)))$ where S is a non-central proper subgroup of E, and E is chosen as in the case of the rational class functions. With this notation, we can prove the following theorem.

Theorem 5.1. *Let P be a p-group of rank at least 2. There is an exact sequence of abelian groups*

$$0 \longrightarrow C_b(P) \xrightarrow{r_P} \varprojlim_{1<Q\leq P} C_b(N_P(Q)/Q) \xrightarrow{\tilde{d}_P} \tilde{H}^0_b(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P \longrightarrow 0$$

where the maps r_P and $\tilde{d}_P$ are as defined above.

Proof. The proof of the injectivity of r_P is similar to the proof of the injectivity of $r_P^{\mathcal{R}^*_{\mathbb{Q}}}$. To prove that $\tilde{d}_P$ is surjective, note that we can embed $\tilde{H}^0_b(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P$ into $\tilde{H}^0(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P$. Now if $n_E\chi_E$ is the characteristic function of the class of the maximal elementary abelian subgroup E of P of rank 2, then we regard $pn_E\chi_E$ as a rational class function and using the surjectivity of $\tilde{d}_P^{\mathcal{R}^*_{\mathbb{Q}}}$, we construct the pre-image $(f_Q)_{1<Q\leq P}$ of $n_E\chi_E$ in $\varprojlim_{1<Q\leq P} \mathcal{R}^*_{\mathbb{Q}}(N_P(Q)/Q)$. Now this class is a gluing data for Borel-Smith functions since f_Q is non-zero only if Q is P-conjugate to a non-central proper subgroup of E in which case it is either 1 or 2 or 4 according to whether $C_P(Q)/Q$ is cyclic of order 2 or cyclic of order at least 3 or generalized

quaternion and in this case it is clear that the function f_Q is a Borel-Smith function, proving the surjectivity of the map $\tilde{d}_P$.

Finally, the proof of the exactness at the middle term of the sequence is similar to the proof of the exactness at the middle term of the sequence of Theorem 4.4. □

As in the previous case for rational class functions, if the poset $\mathcal{A}_{\geq 2}(P)$ is connected, the group $H^0_b(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P$ vanishes and we obtain the following corollary.

Corollary 5.2. *Let P be a p-group such that the poset $\mathcal{A}_{\geq 2}(P)$ is connected. Then the map*

$$r_P^{C_b} : C_b(P) \to \varprojlim_{1<Q\leq P} C_b(N_P(Q)/Q)$$

is an isomorphism.

Finally, the following corollary completes the solution of the gluing problem for Borel-Smith functions for p-groups. We leave the straightforward proof as an exercise.

Corollary 5.3. *Let P be a p-group of rank 1. Then there is an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow C_b(P) \xrightarrow{r_P} \varprojlim_{1<Q\leq P} C_b(N_P(Q)/Q) \longrightarrow \mathbb{Z}/m_P\mathbb{Z} \longrightarrow 0$$

of abelian groups where $m_P = 2$ if P is cyclic and $m_P = 4$ if P is generalized quaternion.

6. The Main Result

The main result of the paper is the following theorem which relates the endo-trivial group to the group of obstructions of the gluing problem for the functor of Borel-Smith functions. As an immediate corollary, we will obtain a set of generators for the group $T(P)$ of endo-trivial modules.

Theorem 6.1. *Let p be a prime number and P be a p-group, which is not cyclic, quaternion or semi-dihedral. There is a split exact sequence of abelian groups*

$$0 \longrightarrow \mathbb{Z} \longrightarrow T(P) \longrightarrow \mathrm{Obs}(C_b(P)) \longrightarrow 0$$

where $T(P)$ is the group of endo-trivial modules and $\mathrm{Obs}(C_b(P))$ is the group of obstructions for the gluing problem at P for the biset functor of Borel-Smith functions.

Proof. The proof consists of two parts. The first part is the proof of the existence of the exact sequence and the second part is the construction of an explicit splitting.

For the first part of the proof, recall that, by Theorem 1.2 of [8], there is an exact sequence of p-biset functors

$$0 \longrightarrow C_b \longrightarrow B^* \xrightarrow{\Psi} D^\Omega \longrightarrow 0.$$

Here the map Ψ is defined as the family (Ψ_P) where Ψ_P maps an element $\omega_{P/Q}$ of the basis $\{\omega_{P/Q} | Q \leq_P P\}$ of $B^*(P)$, introduced in Lemma 2.2 of [4], to the corresponding relative syzygy $\Omega_{P/Q}$.

Now by Proposition 2.1, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_b(P) \arrow[r] \arrow[d, "r_P^{C_b}"'] & B^*(P) \arrow[r, "\Psi_P"] \arrow[d, "r_P^{B^*}"'] & D^\Omega(P) \arrow[r] \arrow[d, "r_P^D"'] & 0 \\
0 \arrow[r] & \varprojlim_{1<Q\le P} C_b(W_P(Q)) \arrow[r] & \varprojlim_{1<Q\le P} B^*(W_P(Q)) \arrow[r, "\tilde\Psi_P"] & \varprojlim_{1<Q\le P} D^\Omega(W_P(Q)) &
\end{tikzcd}$$

where the maps $\tilde{\Psi}_P$ and r_P's are defined as in Section 2. Now by Snake Lemma together with Theorem 3.1, we obtain an exact sequence

$$0 \longrightarrow Ker(r_P^{C_b}) \longrightarrow \mathbb{Z} \longrightarrow T^\Omega(P) \longrightarrow \mathrm{Obs}(C_b(P)) \longrightarrow 0$$

where $T^\Omega(P)$ denotes the kernel of $r_P^{D^\Omega}$. Now we claim that the equality

$$T^\Omega(P) = T(P)$$

holds. Indeed, by [17, Section 2.1.1], the kernel of r_P^D is equal to the group $T(P)$ and hence we have

$$T^\Omega(P) = T(P) \cap D^\Omega(P).$$

But by Theorem 7.7 in [4], any class in the Dade group which is not equal to a class of relative syzygy has finite order, and by Theorem 1.1 in [11], the group $T(P)$ is torsion free if P is not cyclic, generalized quaternion or semi-dihedral. Thence, we have $T^\Omega(P) = T(P)$.

On the other hand, by the proof of Theorem 3.1, the group $\mathbb{Z}$, in the above sequence, can be identified with the subgroup of $B^*(P)$ generated by ω_P. Now by the definition of the map Ψ, the image of ω_P is equal to Ω_P. Moreover it is well-known that the subgroup of $T(P)$ generated by Ω_P is infinite cyclic since P is not cyclic or quaternion, see [14], cf. [18]. Therefore we get that $\mathrm{Ker}(r_P^{C_b}(P)) = 0$, which completes the first part of the proof of the theorem.

For the second part of the proof, first note that the surjective map $T(P) \to \mathrm{Obs}(C_b(P))$ is the connecting homomorphism of the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & T(P) \arrow[d] & \\
0 \arrow[r] & C_b(P) \arrow[r] \arrow[d, "r_P"'] & B^*(P) \arrow[r, "\Psi_P"] \arrow[d, "r_P"'] & D^\Omega(P) \arrow[r] \arrow[d, "r_P"'] & 0 \\
0 \arrow[r] & \varprojlim_{1<Q\le P} C_b(W_P(Q)) \arrow[r] \arrow[d] & \varprojlim_{1<Q\le P} B^*(W_P(Q)) \arrow[r, "\tilde\Psi_P"] \arrow[d] & \varprojlim_{1<Q\le P} D^\Omega(W_P(Q)) & \\
 & \mathrm{Obs}(C_b(P)) \arrow[r] & 0 & &
\end{tikzcd}$$

To be able to construct a splitting for this connecting homomorphism, notice that by Theorem 5.1, we also have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_b(P) \arrow[r] \arrow[d] & \mathcal{R}^*_{\mathbb{Q}}(P) \arrow[r, "\Psi_P"] \arrow[d] & D^{\Omega}_t(P) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim_{1<Q\le P} C_b(W_P(Q)) \arrow[r] \arrow[d] & \varprojlim_{1<Q\le P} \mathcal{R}^*_{\mathbb{Q}}(W_P(Q)) \arrow[r] \arrow[d] & \varprojlim_{1<Q\le P} D^{\Omega}_t(W_P(Q)) \arrow[d] & \\
0 \arrow[r] & \tilde{H}^0_b(\mathcal{A}_{\ge 2}(P), p\mathbb{Z})^P \arrow[r] \arrow[d] & \tilde{H}^0(\mathcal{A}_{\ge 2}(P), p\mathbb{Z})^P \arrow[r] \arrow[d] & \mathrm{Obs}(D^{\Omega}_t) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and the identification

$$\mathrm{Obs}(C_b(P)) = \tilde{H}^0_b(\mathcal{A}_{\ge 2}(P), p\mathbb{Z})^P.$$

Here, as usual, $\hookrightarrow$ denotes an injective map and $\twoheadrightarrow$ denotes a surjective map.

Now the splitting is defined in the following way. First consider the third diagram in this proof. Let $pn_E\chi_E \in \tilde{H}^0_b(\mathcal{A}_{\ge 2}(P), p\mathbb{Z})^P$ be a generator of this group where E is a maximal elementary abelian subgroup E of P of rank 2. We regard it as an element in $\tilde{H}^0(\mathcal{A}_{\ge 2}(P), p\mathbb{Z})^P$. By the proof of Theorem 4.4, the pre-image of $pn_E\chi_E$ in $\varprojlim_{1<Q\le P} \mathcal{R}^*_{\mathbb{Q}}(W_P(Q))$ is the sequence $(f_Q)_{1<Q\le P}$ given by $f_Q = 0$ unless Q is conjugate to a non-central cyclic subgroup, say A, of E in which case f_Q is given by $f_Q(T/Q) = n_E\delta_{T,Q}$. Since it is not zero, the gluing data $(f_Q)_{1<Q\le P}$ is not coming from a rational class function.

But we pass to the second diagram of this proof by embedding this data to the group of the gluing data of super class functions. Then by Theorem 3.1, this data can be glued to a super class function, say f, which is given by, $f(P/Q) = 0$ unless Q is conjugate to A in which case $f(P/Q) = n_E$. Using the basis $\{\omega_{P/Q} | Q \le_P P\}$ of $B^*(P)$, we can write

$$f = n_E(\omega_{P/A} - \omega_{P/1}).$$

Now by the definition of the map Ψ^P, we have

$$\Psi^P(f) = [(\Omega_{P/A} \otimes \Omega^{-1}_{P/1})^{\otimes n_E}]$$

where for an endo-permutation module M, we denote its class in the Dade group by $[M]$. Hence the above construction gives a homomorphism

$$\sigma : H^0_b(\mathcal{A}_{\ge 2}(P), p\mathbb{Z})^P \to D(P).$$

of abelian groups. Now we claim that the class of $(\Omega_{P/A} \otimes \Omega^{-1}_{P/1})^{\otimes n_E}$ in the Dade group $D(P)$ is in the image of the endo-trivial group. Remark that this class coincides with the class found by Alperin in [1] where he proved this claim. However we can give a direct proof of the claim.

To prove the claim, we need to prove that the class $(\Omega_{P/A} \otimes \Omega^{-1}_{P/1})^{\otimes n_E}$ is in the kernel of the map r^D_P but this is trivial since the gluing data $(f_Q)_{1<Q\le P}$ is a data

for Borel-Smith functions, by its construction and hence the claim follows from the commutativity of the second diagram of this proof.

Hence we get that the class $[(\Omega_{P/A}\otimes\Omega_{P/1}^{-1})^{\otimes^{n_E}}]$ is in the image of $T(P)$ in $D(P)$ which gives a homomorphism

$$\sigma : H_b^0(\mathcal{A}_{\geq 2}(P), p\mathbb{Z})^P \to T(P).$$

of abelian groups. To end the proof, we need to check that this map is a splitting for the connecting homomorphism but this is straightforward from the construction. □

The following theorem follows from the above proof. Note that although we end up with the same set of generators for the endo-trivial group found by Alperin in [1], our proof does not depend on his result, hence gives another proof of the classification theorem of Carlson and Thévenaz in [10].

Theorem 6.2 (Carlson-Thévenaz[10])**.** *Let $p > 2$ be a prime number and P be a non-abelian p-group which is not semi-dihedral. Then the group $T(P)$ of endo-trivial modules for P is free abelian on the basis $\Omega_P, \Gamma((\Omega_{P/A}\otimes\Omega_{P/1}^{-1})^{\otimes^{n_E}})$ as E runs over a complete set of representatives of conjugacy classes of maximal elementary abelian subgroups of P of rank 2. Here, for a kP-module M, the module $\Gamma(M)$ is the sum of all indecomposable summands of M with vertex P.*

We end the paper by noting that our proof does not use the classification theorem of Carlson and Thévenaz, (which means that our arguments are not circular). Indeed, we have used three results from general theory of endo-permutation modules. Namely, the theorems about two exact sequences from [8], Theorem 7.1 from [4] which states that the classes of endo-permutation modules which are not relative syzygies are torsion and the injectivity of the map $r_P^{D_t}$ from [7]. Finally we have used the description of the torsion part of the group $T(P)$ from [11]. A careful reading would show that all these results together with any result that they refer are independent of this classification theorem.

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